

Chapter 8

A Stochastic Description of Eddy–Mean Flow Interactions



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Abstract Interactions between the mean flow and eddies play a central role in shaping oceanic circulation by redistributing heat, momentum, and energy. Accurately representing these processes in General Circulation Models (GCMs) remains a challenge due to limitations in subgrid-scale parameterizations. This study highlights the constraints imposed by small ensemble sizes on capturing the Reynolds stress tensor and diagnosing its associated work—key elements for assessing mean-to-eddy energy conversion rates. To address these challenges, we employ the Location Uncertainty (LU) framework, which introduces stochastic variability into the governing equations to represent unresolved turbulent effects. Using a 48-member ensemble simulation of the North Atlantic under realistic forcing, we compare deterministic and stochastic estimates of Reynolds stress work. The LU framework improves statistical convergence and stabilizes higher-order moments, leading to a more accurate representation of energy transfer statistics. Key contributions include a novel formulation of energy transfer equations and enhanced statistical diagnostics of Reynolds stress work within the LU framework. This study provides valuable dynamical diagnostics for identifying energetic patterns and regions of stability, offering new insights into eddy–mean flow interactions in the Gulf Stream.

8.1 Introduction

General Circulation Models (GCMs) are essential tools for studying oceanic and climate systems, enabling the simulation and analysis of complex geophysical interactions. The MIT General Circulation Model (MITgcm) (Marshall et al. 1997), used

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in this study, is specifically designed to capture large-scale oceanic and atmospheric dynamics. GCMs play a crucial role in predicting climate variability, analyzing ocean currents, and assessing the impact of physical processes on the Earth system.

A key advancement in modern GCM applications is the use of ensemble simulations to account for the intrinsic variability and chaotic nature of geophysical flows (Palmer 2012; Leutbecher and Palmer 2008). Due to the sensitivity of such systems to initial conditions, even small perturbations can lead to markedly different outcomes. Ensemble simulations address this by running multiple instances of the model with slightly varied initial states, enabling a probabilistic characterization of system behavior. This approach facilitates the exploration of a range of possible outcomes and allows for the estimation of statistical moments without relying on ergodic assumptions. Ensembles are particularly valuable for quantifying uncertainties, assessing the likelihood of extreme events, and evaluating the robustness of climate projections. However, to ensure reliable interpretation, the empirical moments computed from ensemble simulations must be statistically converged. Without sufficient convergence, these estimates become unreliable and can lead to misleading conclusions. Achieving statistical convergence typically requires large ensemble sizes, which, in high-dimensional systems such as GCMs, can result in prohibitive computational costs.

Another major challenge in geophysical fluid simulations is the accurate representation of unresolved subgrid-scale (SGS) processes, due to the vast range of spatial and temporal scales involved. Resolving all relevant scales is computationally infeasible, necessitating the use of SGS parameterizations to model the influence of smaller, unresolved motions on the resolved flow. Central to these parameterizations is the Reynolds stress tensor, $\tau = \overline{\mathbf{u}' \otimes \mathbf{u}'}$, which captures the covariance of velocity fluctuations and their associated momentum transfer. Here, the $\otimes$ notation denotes the outer product of two vectors and the overbar represents a chosen averaging operator. This tensor plays a key role in governing how unresolved turbulence influences larger-scale dynamics, particularly in mediating energy transfers across scales. However, in practical applications, the Reynolds stress tensor is not directly available and must be approximated.

A common approach to approximating the Reynolds stress tensor involves mixing-length arguments, which relate it to an eddy viscosity coefficient intended to mimic the dissipative effects of turbulence. In its simplest form, eddy viscosity is prescribed constant in both time and space, typically tuned to ensure numerical stability (e.g., Adcroft 1995). This approach underpins widely used subgrid-scale parameterizations, such as those developed in Redi (1982), Gent and McWilliams (1990). More recent developments have focused on spatially and temporally varying eddy viscosities, as well as on incorporating energy backscatter mechanisms to better capture the bidirectional nature of energy transfer and improve physical realism (Bachman 2019; Eden and Greatbatch 2008; Jansen et al. 2019; Marshall et al. 2012). Evaluating the performance of SGS closures typically relies on reference simulations, as in Eden and Greatbatch (2008), where a non-eddy-resolving simulation employing an active SGS closure is compared against a high-resolution twin simulation (with the SGS closure disabled), filtered onto the coarse grid. While effective, this strategy requires

to predefine the scale of the “eddies” that have to be parameterized. From a physical perspective, it is often challenging to disentangle the effects of genuinely unresolved turbulent scales from those of marginally resolved coherent structures. The latter may violate the assumptions of homogeneity and isotropy that underlie many SGS models, potentially introducing non-local effects that standard parameterizations fail to capture (Grooms et al. 2013; Kang and Curchitser 2015).

To address the limitations of spatial filtering diagnostics, recent work has introduced ensemble-based approaches that decompose the flow into an ensemble mean and fluctuations about that mean (Jamet et al. 2022; Li et al. 2023; Uchida et al. 2021, 2022). In this framework, the ensemble mean represents the component of the flow that is coherent across realizations and responds to common external forcing, while the deviations from this mean capture the stochastic, realization-specific components associated with nonlinear turbulent dynamics—commonly referred to as “eddies.” From a parameterization perspective, both the ensemble mean and the eddy components possess distinct spatial and temporal scales that can be identified a posteriori (e.g., Jamet et al. 2022) to inform the construction of subgrid-scale models. Although the ensemble mean often exhibits larger-scale structures than the eddies, leading to a common—albeit somewhat imprecise—interpretation of the former as the “large-scale flow” and the latter as the “small-scale” or turbulent component. This ensemble-based decomposition enables the estimation of turbulent statistics without assuming spatial homogeneity, isotropy, or temporal stationarity. The strength of this approach lies in its generality and flexibility. However, as discussed above, generating large ensembles is computationally expensive. Most practical ensembles contain fewer than 100 members, which limits statistical convergence—particularly given the slow convergence rate of Monte Carlo methods, which scales as $1/\sqrt{n}$. For instance, achieving a precision of 1% would require approximately 10,000 samples.

In this study, we propose a novel approach to define robust statistical diagnostics from small ensembles in high-dimensional ocean simulations. Our method leverages the Location Uncertainty (LU) framework (Mémin 2014), which derives stochastic fluid equations via a probabilistic reformulation of the Reynolds transport theorem and fundamental conservation laws (Bauer et al. 2020a; Chapron et al. 2018; Resseguier et al. 2017a). By interpreting ensemble fluctuations as realizations of stochastic fields, the LU framework enables the generation of an arbitrary number of virtual samples for improved statistical estimation. We demonstrate the effectiveness of this framework using a 48-member eddy-resolving ensemble (at $1/12^\circ$ resolution) of the Gulf Stream.

The paper is structured as follows: Sect. 8.2 presents the theoretical background for representing kinetic energy exchanges between the mean flow and eddies within an ensemble-averaged framework, followed by an overview of the LU formulation. Section 8.3 provides numerical results illustrating the improved stability of energy diagnostics achieved with the LU approach, alongside comparisons with traditional empirical estimates. Finally, Sect. 8.4 offers concluding remarks and outlines directions for future research.

8.2 Background and Methods

8.2.1 Model

This study is based on eddy-resolving ensemble simulations of the North Atlantic Basin, obtained by numerically integrating the hydrostatic primitive equations (HPE). Details of the numerical code and model configuration are provided in Sect. 8.3. The horizontal momentum equations of the HPE system are given by

$$\begin{cases} \partial_t \mathbf{u}_h = -\nabla \cdot (\mathbf{u} \otimes \mathbf{u}_h) - f \mathbf{k} \times \mathbf{u}_h - \frac{1}{\rho_0} \nabla_h P + \mathcal{F} + \mathcal{D}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \quad (8.1)$$

where P is the pressure, f is the Coriolis parameter, and $\rho_0 = 999.8 \text{ kg m}^{-3}$ is the reference seawater density under the Boussinesq approximation. The velocity field is denoted by $\mathbf{u} = (\mathbf{u}_h, w) = (u, v, w)$ the full 3D velocity field, and $\mathbf{u}_h$ its horizontal component. The terms $\mathcal{F}$ and $\mathcal{D}$ represent external forcing and dissipation, respectively. The vertical velocity component w is computed diagnostically from the continuity equation to ensure incompressibility. The full HPE system includes additional thermodynamic equations for temperature T and salinity S , as well as an equation of state, which are omitted here for clarity.

8.2.2 Eddy–Mean Flow Kinetic Energy Transfers

The analysis of interactions between small-scale fluctuations and the mean flow within the kinetic energy budget is central to the parameterization of subgrid-scale (SGS) eddies and the representation of velocity fluctuations. Numerical models are typically characterized by an effective resolution, enabling a decomposition of the flow into large-scale (resolved) and small-scale (unresolved) components—often defined via a wavenumber cutoff.

Oceanic flow energy can be decomposed into kinetic energy and potential energy. In stratified flows, potential energy is associated with buoyancy forces. Although both total and available potential energies are important concepts, we restrict our attention here to the horizontal kinetic energy:

$$K = \frac{\rho_0}{2} \mathbf{u}_h \cdot \mathbf{u}_h = \frac{\rho_0}{2} (u^2 + v^2), \quad (8.2)$$

Applying Reynolds decomposition $\mathbf{u}_h = \overline{\mathbf{u}_h} + \mathbf{u}_h'$ —where $\overline{\cdot}$ denotes an averaging operator, left unspecified for now, and $\cdot'$ the deviation from the mean—yields

$$K = \underbrace{\frac{\rho_0}{2} \overline{\mathbf{u}_h \cdot \mathbf{u}_h}}_{\text{Mean kinetic energy } K_M} + \underbrace{\frac{\rho_0}{2} \overline{\mathbf{u}'_h \cdot \mathbf{u}_h'}}_{\text{Eddy kinetic energy } K_E} + \underbrace{\rho_0 \overline{\mathbf{u}_h \cdot \mathbf{u}_h'}}_{\text{Cross term } K_C}. \quad (8.3)$$

In this study, we are primarily interested in the evolution of the mean kinetic energy (MKE), K_M . A budget for K_M can be derived by taking the dot product of the averaged momentum equation (8.1) with the mean velocity $\rho_0 \overline{\mathbf{u}_h}$:

$$\begin{aligned} \rho_0 \overline{\mathbf{u}_h} \cdot \overline{\partial_t \mathbf{u}_h} &= \rho_0 \overline{\mathbf{u}_h} \cdot \overline{\left(-\nabla \cdot (\mathbf{u} \otimes \mathbf{u}_h) - f \mathbf{k} \times \mathbf{u}_h - \frac{1}{\rho_0} \nabla_h P \right)}, \\ \partial_t K_M &= -\nabla \cdot (\overline{\mathbf{u}} K_M) - \rho_0 \overline{\mathbf{u}_h} \cdot \nabla \cdot (\overline{\mathbf{u}' \otimes \mathbf{u}'_h}) - \nabla \cdot (\overline{\mathbf{u}} \overline{P}) - \overline{w \bar{b}}. \end{aligned} \quad (8.4)$$

The pressure term transforms as follows:

$$-\overline{\mathbf{u}_h} \cdot \nabla_h \overline{P} = -\nabla \cdot (\overline{\mathbf{u}} \overline{P}) - \overline{w \bar{b}}, \quad (8.5)$$

using incompressibility and the hydrostatic relation $\bar{b} = -\partial_z \overline{P}$. This yields the following form of the MKE budget:

$$\partial_t K_M = -\nabla \cdot (\overline{\mathbf{u}} K_M) - \nabla \cdot (\overline{\mathbf{u}} \overline{P}) - \overline{w \bar{b}} - \underbrace{\rho_0 \overline{\mathbf{u}_h} \cdot \nabla \cdot (\overline{\mathbf{u}' \otimes \mathbf{u}'_h})}_{\text{MEC}}. \quad (8.6)$$

The first two terms on the right-hand side of Eq. (8.6) represent the divergence of kinetic energy and pressure fluxes, respectively. The third term, involving $\overline{w \bar{b}}$, captures the conversion between kinetic and potential energies, often referred to as buoyancy work. The final term, labeled MEC (mean-to-eddy conversion), quantifies the interaction between the mean flow and the Reynolds stress tensor. It represents the rate at which kinetic energy is transferred from the mean flow to the eddies. This term is also commonly referred to as the barotropic energy conversion (BTR) in the literature (see, e.g., Matsuta and Masumoto 2021).

The MEC term plays a central role in the development and evaluation of subgrid-scale (SGS) parameterizations. When using a spatial filtering approach (e.g., Aluie et al. 2018; Grooms et al. 2021), the averaging operator separates the flow into resolved (mean) and unresolved (eddy) components. In this context, K_M corresponds to the kinetic energy of the resolved flow, while $\overline{K_E}$ represents the kinetic energy associated with unresolved fluctuations. The MEC term quantifies the energy exchange between these scales, and its accurate estimation is therefore essential for properly capturing the dynamical influence of unresolved turbulence on the resolved flow.

We propose to diagnose the MEC term using an ensemble-based approach, in which the averaging operator is defined as an ensemble mean, and perturbations are interpreted as deviations of each ensemble member from this mean. This removes the need to specify an explicit spatial or temporal cutoff scale, enabling a more flexible and flow-driven characterization of the dynamics. As a result, the ensemble-based

framework provides a robust means of diagnosing the spatiotemporal structure of the MEC term across a range of dynamical regimes.

A major limitation of the ensemble-based approach is the typically small number of available realizations. While ensemble analysis holds great promise for diagnosing scale interactions and informing subgrid-scale modeling, assembling sufficiently large ensembles remains computationally expensive. With a limited ensemble size, the empirical mean may converge poorly, leading to biased or noisy estimates of the Reynolds stress tensor. As we will show, the available dataset with $N = 48$ members does not achieve full convergence. As a result, variability across realizations is underrepresented, limiting the robustness of diagnostics based on the ensemble-averaged MEC.

8.2.3 Location Uncertainty (LU)

In the following sections, we introduce an alternative approach to better capture eddy–mean flow interactions, based on the stochastic framework of Location Uncertainty (LU) (Mémín 2014). This framework models velocity fluctuations as random variables, assuming they decorrelate over the characteristic timescale of the large-scale flow. This mathematical formulation allows LU to capture flow variability and the complex dynamics of eddy–mean flow interactions from a stochastic standpoint. As we will show, it enables the computation of stable statistical diagnostics using the same ensemble of simulations.

8.2.3.1 Particle Displacement

Location Uncertainty (LU) is a framework that reformulates the equations of fluid mechanics by incorporating stochastic variables and processes directly into their structure (see, for example, Bauer et al. 2020b,a; Brecht et al. 2021; Li et al. 2023; Resseguier et al. 2017a,b,c; Tucciarone et al. 2023). The core concept of LU is to augment the Lagrangian displacement of a fluid particle with a stochastic component that represents positional “uncertainty” or “fluctuation.” Specifically, the displacement dX_t is decomposed into a smooth-in-time velocity part, $\mathbf{u}$, and a stochastic component that is correlated in space, but decorrelated in time:

$$\mathbf{X}_t = \mathbf{X}_{t_0} + \int_{t_0}^t \mathbf{u}(\mathbf{X}_s, s) ds + \int_{t_0}^t \boldsymbol{\sigma}_s d\mathbf{B}_s, \quad (8.7)$$

where the stochastic integral in the last term is understood in the Itô sense. In differential form, this reads

$$d\mathbf{X}_t = \mathbf{u}_t dt + \boldsymbol{\sigma}_t d\mathbf{B}_t. \quad (8.8)$$

Here, the first term represents the smooth deterministic displacement, while the second term, $\sigma_t d\mathbf{B}_t$, corresponds to the stochastic displacement accounting for unresolved turbulent effects. This stochastic component will henceforth be referred to as the “noise.”

8.2.3.2 Noise Term $\sigma d\mathbf{B}_t$

The term $\sigma d\mathbf{B}_t$ represents small-scale uncertainty, encapsulating the spatially correlated variability of the flow. It is a vector with one component per spatial dimension, defined as

$$(\sigma_t d\mathbf{B}_t)^{(i)}(\mathbf{x}) = \int_{\Omega} \check{\sigma}_{ik} d\mathbf{B}_t^k(\mathbf{x}, \mathbf{x}', t) d\mathbf{x}', \quad (8.9)$$

where the Einstein summation convention over repeated indices is adopted. Here, σ acts as an integral operator on a well-defined functional Brownian motion $d\mathbf{B}_t$, with the matrix-valued kernel $\check{\sigma}(\mathbf{x}, \mathbf{x}', t)$ introducing spatial correlations that maintain coherence between points $\mathbf{x}$ and $\mathbf{x}'$. This formulation allows for an effective statistical representation of unresolved small-scale flows.

The kernel $\check{\sigma}$ is assumed symmetric, positive-definite, and bounded in space and time. For simplicity, we consider a deterministic kernel here, though random kernels can also be incorporated (Li et al. 2023). Under these assumptions, the correlation operator is a positive-definite Hilbert–Schmidt operator. It admits a spectral decomposition into a set of eigenfunctions φ_n , $n \in \mathbb{N}$, with corresponding eigenvalues λ_n , forming an orthonormal basis. Consequently, the noise can be expressed as

$$\sigma_t d\mathbf{B}_t(\mathbf{x}) = \sum_{n \in \mathbb{N}} \sqrt{\lambda_n} \varphi_n(\mathbf{x}, t) d\beta_n, \quad (8.10)$$

where $(\beta_n)_{n \in \mathbb{N}}$ are independent standard Brownian motions.

8.2.3.3 Covariance Tensor

By construction, the noise term $\sigma_t d\mathbf{B}_t$ is a centered Gaussian process whose covariance depends on spatial position and the squared kernel. Using Itô isometry, the covariance between two points $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\text{Cov}((\sigma d\mathbf{B}_t)^{(i)}(\mathbf{x}, t), (\sigma d\mathbf{B}_t)^{(j)}(\mathbf{y}, t')) = q_{ij}(\mathbf{x}, \mathbf{y}, t) \delta(t - t') dt, \quad (8.11)$$

where q_{ij} denotes the (i, j) -th component of the two-point covariance tensor $\mathbf{Q}(\mathbf{x}, \mathbf{y}, t)$, and δ is the Dirac delta function. The one-point covariance tensor is defined as

$$\mathbf{a}(\mathbf{x}, t) dt := \mathbf{Q}(\mathbf{x}, \mathbf{x}, t), \quad (8.12)$$

which represents a generalized eddy viscosity tensor associated with the unresolved small-scale flow component. It has units of m^2/s and, up to a decorrelation time τ , characterizes the variance of the noise. By slight abuse of terminology, $\mathbf{a}$ is often called the variance tensor of the noise. Its trace integrated over the domain Ω provides a bounded measure of the turbulent kinetic energy:

$$\frac{1}{2} \int_{\Omega} \text{Tr} \mathbf{a}(\mathbf{x}, t) d\mathbf{x} < +\infty. \quad (8.13)$$

This expression can also be formulated in terms of the eigenpairs of the correlation tensor:

$$\int_{\Omega} \text{Tr} \mathbf{a}(\mathbf{x}, t) = \sum_{i \in \mathbb{N}} \text{Tr}(\varphi_i(\mathbf{x}) \varphi_i(\mathbf{x})^T) d\mathbf{x}, \quad (8.14)$$

where superscript $\cdot^T$ denotes vector transposition.

8.2.3.4 Transport Operator

The derivation of general stochastic flow dynamics within the LU framework is based on a stochastic extension of the Reynolds transport theorem (Mémín 2014). This theorem allows differentiation of a quantity within a volume transported by the stochastic flow and introduces a stochastic transport operator, which can be interpreted as a stochastic material derivative acting on a transported scalar field θ . This operator is defined as

$$\mathbb{D}_t \theta = \underbrace{d_t \theta}_{\theta(\mathbf{x}, t+dt) - \theta(\mathbf{x}, t)} + \underbrace{(\mathbf{u}^* dt + \boldsymbol{\sigma}_t d\mathbf{B}_t) \cdot \nabla \theta}_{\text{Stochastic advection}} - \underbrace{\frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \theta) dt}_{\text{Stochastic diffusion}}. \quad (8.15)$$

This material derivative comprises three terms. The first term represents the time increment of θ at a fixed location. The second term is the stochastic advection, involving both the noise term and a modified large-scale drift velocity $\mathbf{u}^*$. This drift includes a component $\mathbf{u}_S$ that can be interpreted as a generalization of the Stokes drift associated with orbital wave motion (Bauer et al. 2020a). For incompressible noise (i.e., when $\nabla \cdot \boldsymbol{\sigma} = 0$), it takes the form:

$$\mathbf{u}^* = \mathbf{u} - \mathbf{u}_S = \frac{1}{2} \nabla \cdot \mathbf{a}. \quad (8.16)$$

The Stokes drift $\mathbf{u}_S$ is defined by the divergence of the variance tensor $\mathbf{a}$. This tensor also appears in the last term of Eq. (8.15), which represents stochastic diffusion. It defines the principal axes of noise-induced diffusion (Resseguier et al. 2017d).

8.2.4 LU HPE Horizontal Momentum Equations

Using the stochastic transport operator, any fluid dynamics model can be derived by following the same procedure and approximations as in the deterministic setting. The horizontal momentum equations of the hydrostatic LU model are given by

$$\mathbb{D}_t \mathbf{u}_h = -f \mathbf{k} \times (\mathbf{u}_h dt + \boldsymbol{\sigma}_t d\mathbf{B}_t) - \frac{1}{\rho_0} \nabla dp, \quad (8.17)$$

with pressure expressed as $dp = Pdt + dP^\sigma$, where the last martingale pressure term satisfies $dP^\sigma = 0$. For interested readers, the full set of stochastic primitive equations is provided in Tucciarone et al. (2023).

8.2.4.1 Stochastic Kinetic Energy Equation

To derive the kinetic energy equation within the LU framework, we follow a procedure analogous to the deterministic case. Specifically, we multiply the mean of the stochastic momentum equation (8.17) by the mean flow velocity, i.e., $\rho_0 \overline{\mathbf{u}_h} \cdot \mathbb{D}_t \mathbf{u}_h$. After several derivation steps—detailed in Appendix 8.4—we obtain the following equation for the mean kinetic energy K_M :

$$\partial_t K_M = -\nabla \cdot (\overline{\mathbf{u}} K_M) - \nabla \cdot (\overline{\mathbf{u}} \overline{P}) - \overline{w} \overline{b} + \nabla \cdot (\mathbf{u}_s K_M) + \rho_0 \overline{\mathbf{u}_h} \cdot \left(\frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \overline{\mathbf{u}_h}) \right). \quad (8.18)$$

The deterministic equation (8.6) and its stochastic counterpart (8.18) differ notably in their last two terms. To ensure that both formulations represent the same physical process—albeit with different statistical distributions—we define the stochastic representation of the mean–eddy correction (MEC) by equating these terms, through a specifically calibrated decorrelation time:

$$\underbrace{-\rho_0 \overline{\mathbf{u}_h} \cdot \nabla \cdot (\overline{\mathbf{u}' \otimes \mathbf{u}'})}_{MEC} = \underbrace{\nabla \cdot (\mathbf{u}_s K_M)}_{\text{Advection of } K_M \text{ by } \mathbf{u}_s} + \underbrace{\rho_0 \overline{\mathbf{u}_h} \cdot \left(\frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \overline{\mathbf{u}_h}) \right)}_{\text{Stochastic diffusion}}. \quad (8.19)$$

The right-hand side of this equation includes two notable contributions: the advection of mean kinetic energy by the Itô–Stokes drift, and a diffusion term arising from stochastic transport. In practice, this diffusive contribution can be efficiently approximated using a Milstein-type numerical scheme, as detailed in Appendix 8.4. This approach interprets the diffusion as a double advection of the mean velocity $\mathbf{u}_h$ by the noise $\boldsymbol{\sigma}_t d\mathbf{B}_t$, a term we denote as $DADV$. When averaged over a sufficiently large ensemble, this double advection term converges to the diffusive expression:

$$DADV = \mathbb{E} \left[\rho_0 \overline{\mathbf{u}_h} \cdot \left((\boldsymbol{\sigma}_t d\mathbf{B}_t \cdot \nabla) (\boldsymbol{\sigma}_t d\mathbf{B}_t \cdot \nabla) \overline{\mathbf{u}_h} \right) \right]. \quad (8.20)$$

To evaluate both terms on the right-hand side of (8.19), we need a method to sample the noise term and to compute the variance tensor $\mathbf{a}$, involved both in the diffusion and in the Itô-Stokes drift. We also need a method to specify the decorrelation time to ensure the coherence of the stochastic and deterministic expressions of the mean Kinetic energy (8.6) and (8.19). The following section describes the approaches used for this purpose.

8.2.4.2 Noise Ansatz

As noted earlier, the noise can be effectively constructed using a spectral representation of the correlation tensor. To this end, we define a set of time-varying empirical eigenfunctions $\{\varphi_k(\mathbf{x}, t)\}_{k=1}^m$ and corresponding eigenvalues $\lambda_k(t)$ following the Proper Orthogonal Decomposition (POD) approach of Uchida et al. (2022). At each time t , POD is applied to the ensemble fluctuation matrix

$$\mathbf{U} = [\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_m] \in \mathbb{R}^{n \times m},$$

where each column $\mathbf{u}'_i \in \mathbb{R}^n$ represents the horizontal velocity fluctuation of the i -th ensemble member relative to the ensemble mean. This yields a spectral representation of the noise, along the ensemble dimension m , at fixed time t :

$$\sigma_t \mathrm{d}B_t^x(\mathbf{x}) = \sqrt{\tau} \sum_{k=1}^m \sqrt{\lambda_k(t)} \varphi_k^x(\mathbf{x}, t) \mathrm{d}\beta_t^k, \quad \sigma_t \mathrm{d}B_t^y(\mathbf{x}) = \sqrt{\tau} \sum_{k=1}^m \sqrt{\lambda_k(t)} \varphi_k^y(\mathbf{x}, t) \mathrm{d}\beta_t^k. \quad (8.21)$$

Here, τ denotes the decorrelation time, which ensures that the correlation operator has the correct physical units of $m/\sqrt{s}$ —since the noise represents a displacement and the eigenfunctions are velocity fields. An accurate estimation of τ —discussed in Appendix 8.4—is essential for the noise model to faithfully capture the statistical properties of the underlying velocity fluctuations as well their kinetic energy.

From a practical standpoint, we employ the snapshot POD method (Sirovich 1987), as detailed in Appendix 8.4. To preserve the full variability captured by the ensemble, we retain all $m - 1$ nontrivial eigenfunctions. This spectral representation of the noise provides a flexible basis that enables the fast generation of an arbitrary number of ensemble fluctuation samples.

The top panel of Fig. 8.1 shows a realization of the two horizontal noise components, normalized by $\sqrt{\tau}$ to express them in velocity-like units. The bottom panel displays the corresponding velocity fluctuations. Due to the construction method, the noise fields share structural features with the fluctuations—for instance, horizontal banding in the x -component and vertical banding in the y -component. However, the noise fields exhibit finer-scale variability, leading to less regular and more intricate patterns compared to the smoother, large-scale structure observed in the velocity fluctuations.

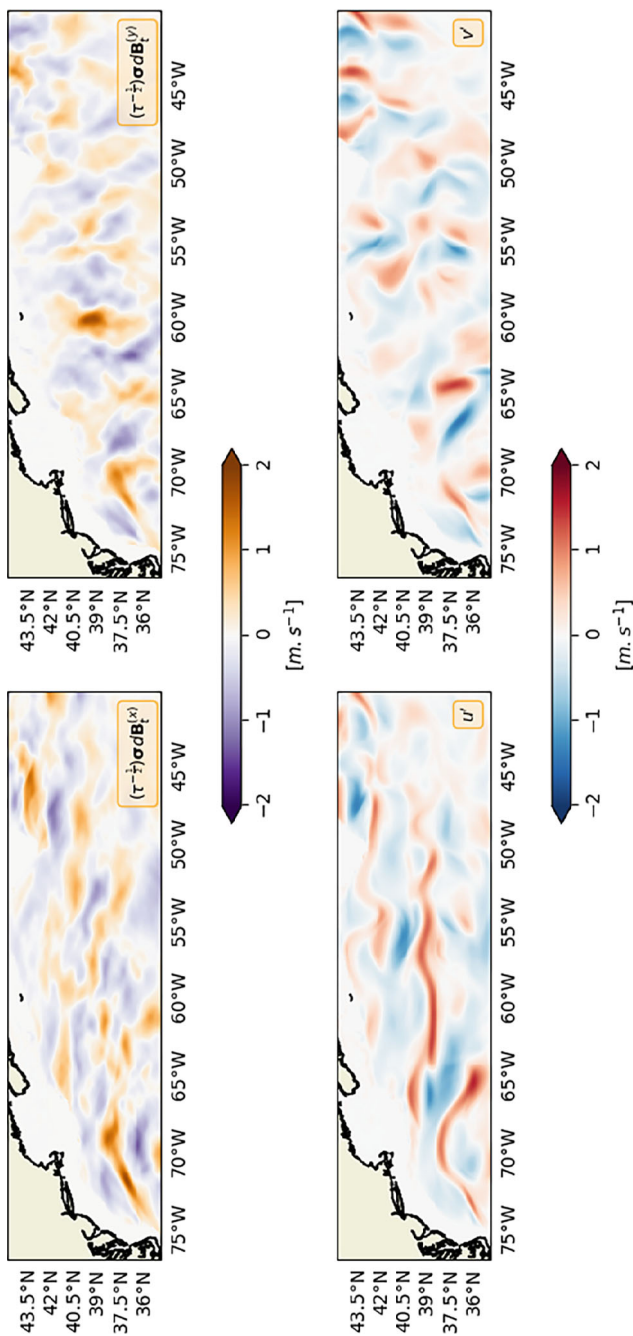


Fig. 8.1 Example of a realization of the x (left) and y (right) components of the noise normalized by $\sqrt{T}$ (top panel), and of the velocity fluctuations u' and v' (bottom panel). All maps are shown at $z = 94$ m on January 1, 2010

The components of the variance tensor can be directly inferred from the spectral decomposition on the m modes via

$$\mathbf{a}^{xx}(\mathbf{x}) = \tau \lambda_k \phi_k^x(\mathbf{x}) \phi_k^x(\mathbf{x}), \quad \mathbf{a}^{xy}(\mathbf{x}) = \tau \lambda_k \phi_k^x(\mathbf{x}) \phi_k^y(\mathbf{x}), \quad \mathbf{a}^{yy}(\mathbf{x}) = \tau \lambda_k \phi_k^y(\mathbf{x}) \phi_k^y(\mathbf{x}), \quad (8.22)$$

where $\mathbf{a}^{xy} = \mathbf{a}^{yx}$ by symmetry.

An example of the variance tensor components is shown in Fig. 8.2. Although $\mathbf{a}^{xy}$ has a smaller amplitude than $\mathbf{a}^{xx}$ and $\mathbf{a}^{yy}$, it is clearly non-negligible, indicating a correlation between the two fluctuation components. Both the noise field and its variance are strongly anisotropic and spatially inhomogeneous. Within the Gulf Stream path, the variance tensor reaches values on the order of $\mathcal{O}(100 \text{ m}^2 \text{ s}^{-1})$, roughly one order of magnitude larger than the explicit horizontal viscosity coefficient $A_h = 20 \text{ m}^2 \text{ s}^{-1}$ used in our $\frac{1}{12}^\circ$ configuration. Nevertheless, these values remain smaller than typical subgrid-scale viscosities in non-eddy-resolving models (i.e., with horizontal resolution $\sim \mathcal{O}(1^\circ)$), where coefficients often reach $\mathcal{O}(1000 \text{ m}^2 \text{ s}^{-1})$. The fact that the dissipation induced by our stochastic representation falls between these two regimes supports the interpretation that the noise effectively models the unresolved eddy feedback on the large-scale mean flow.

8.2.4.3 Statistical Diagnosis

Thanks to the ease of generating samples of the stochastic fluctuations, empirical estimates of the order- p statistical moment of the DADV term can be computed as

$$\mathbb{E}[(DADV - \overline{DADV})^p] \approx \frac{1}{n-1} \sum_{i=1}^n \left[\left(\rho_0 \overline{\mathbf{u}_h} \cdot ((\sigma d \mathbf{B}_t^{(i)} \cdot \nabla)(\sigma d \mathbf{B}_t^{(i)} \cdot \nabla) \overline{\mathbf{u}_h}) - \overline{DADV} \right)^p \right], \quad (8.23)$$

with the mean term defined as

$$\overline{DADV} = \frac{1}{n} \sum_{i=1}^n \left[\rho_0 \overline{\mathbf{u}_h} \cdot ((\sigma d \mathbf{B}_t^{(i)} \cdot \nabla)(\sigma d \mathbf{B}_t^{(i)} \cdot \nabla) \overline{\mathbf{u}_h}) \right], \quad (8.24)$$

where n is the number of realizations. A sufficiently large number of samples ensures statistical convergence. In the next section, we compare these diagnostic quantities to the empirical ensemble-mean estimate of the MEC term.

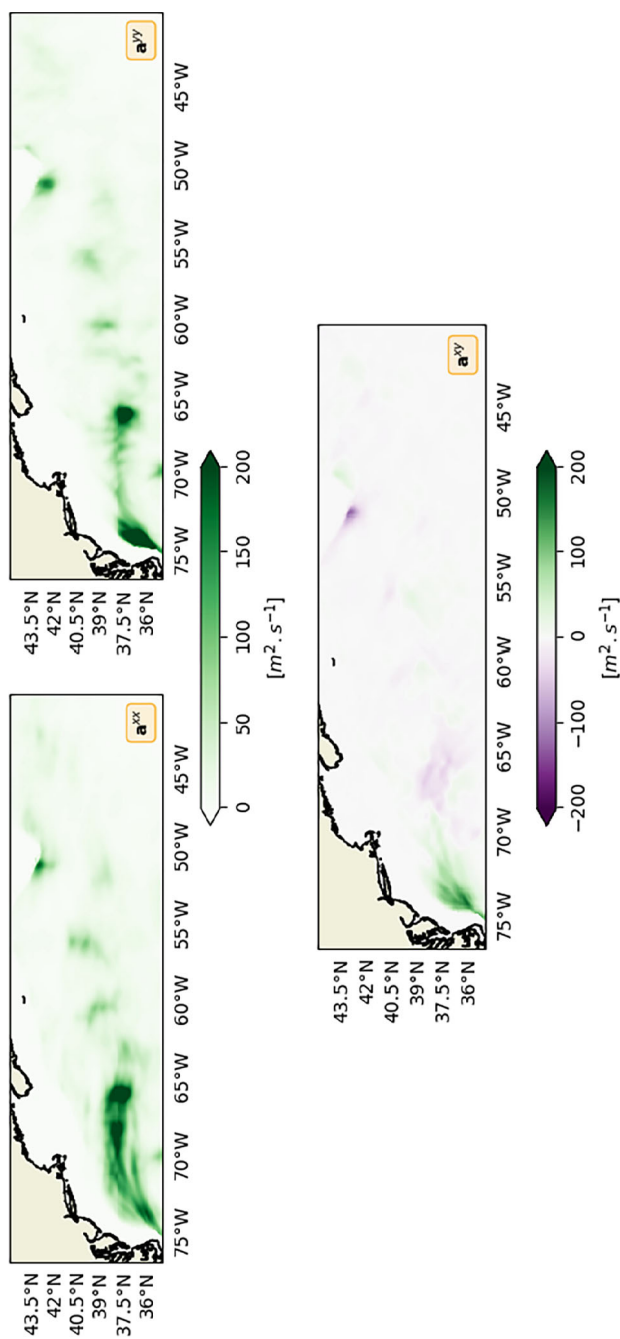


Fig. 8.2 Variance tensor components a^{xx} and a^{xy} (top panel), and a^{xy} (bottom panel), at $z = 94$ m on January 1, 2010

8.3 Numerical Results

8.3.1 Ensemble Simulations

We apply the statistical diagnostics introduced above to an ensemble of simulations generated using the MITgcm ocean model (Marshall et al. 1997), coupled to the idealized atmospheric boundary layer model CheapAML (Deremble et al. 2013). The ensemble consists of 48 eddy-resolving ($1/12^\circ$) simulations of the North Atlantic subtropical gyre, covering the region from 20°S to 55°N . Each simulation spans the period 1963–2012 and is forced by realistic atmospheric and open boundary conditions. Open boundaries at 20°S , 55°N and the Strait of Gibraltar are derived from the ORCA12.L46-MJM88 global ocean configuration based on NEMO (Molines et al. 2014; Sérazin et al. 2015), and linearly interpolated onto the model grid.

At the surface, the ocean is forced by 6-hourly atmospheric winds, precipitation, and radiative fluxes from the Drakkar forcing set (Brodeau et al. 2010; Dussin et al. 2016). Turbulent heat and freshwater exchanges are computed using the COARE3 bulk algorithm (Fairall et al. 2003). CheapAML simulates the horizontal dynamics of air temperature and humidity via a simplified advection–diffusion equation. This setup avoids excessive damping of upper-ocean variability, as compared to more traditional bulk forcing approaches (Jamet et al. 2019). Further documentation and references are available at: <https://github.com/quentinjamet/chaocean>. The 48-member ensemble is built by integrating the model from slightly perturbed initial conditions, following the *micro*-initial condition perturbation framework (Stainforth et al. 2007). This approach captures a range of plausible ocean states under a common external forcing. Unless otherwise noted, diagnostics are computed at $z = -94$ m, within the winter mixed layer (Uchida et al. 2022), and based on snapshot outputs from January 1, 2010. Some diagnostics have also been extended to other times of the year to assess seasonal variability.

In this numerical study, we specifically focus on the deterministic eddy–mean kinetic energy transfer term MEC and its stochastic counterparts: the dissipation term $DADV$ and the Itô–Stokes drift. Figure 8.3 compares the spatial structures of these contributions. The Itô–Stokes drift, driven by the divergence of the noise covariance tensor ($\nabla \cdot \mathbf{a}$), appears negligible in comparison with $DADV$, despite evident spatial heterogeneity in the noise structure (Fig. 8.2). Consequently, this term is not considered further in this study. In contrast, $DADV$ exhibits similar magnitudes and spatial patterns to MEC , supporting its interpretation as a meaningful stochastic proxy for eddy–mean energy exchanges. An illustrative snapshot of the deterministic MEC field on January 1, 2010 is shown in the top panel of Fig. 8.3. It is important to note, however, that unlike $DADV$, the MEC term exhibits poor convergence due to the limited ensemble size, as will be demonstrated. Although MEC is not converged, a qualitative comparison between the two expressions can still be performed.

The structure of the MEC and $DADV$ terms reflects several characteristic patterns of energy transfer. Notably, the Gulf Stream separation acts as a persistent sink of energy for the mean flow, with slowly varying oscillations in the meridional (y)

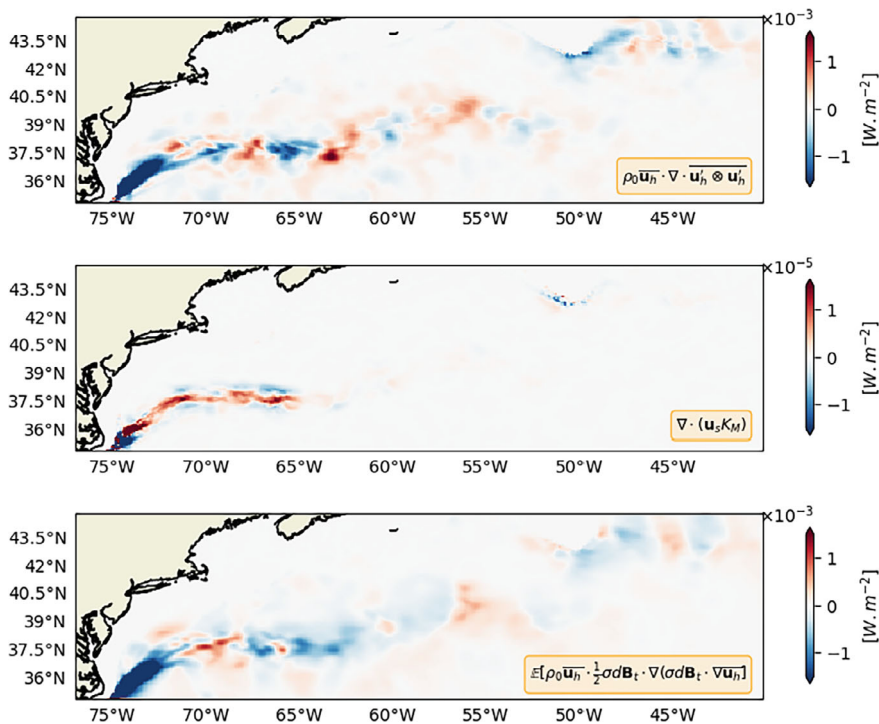


Fig. 8.3 Illustration of the mean–eddy kinetic energy transfer term MEC (upper panel), along with its stochastic equivalent associated with Itô–Stokes drift (center panel) and dissipation ($DADV$, bottom panel). Horizontal maps are taken at $z = 94$ m on January 1, 2010. Note the different scale for the Itô–Stokes drift component

direction. Along the current path, alternating bands of negative and positive energy transfer also appear, indicating a complex spatial organization of eddy–mean interactions. As indicated previously, to ensure consistency between the deterministic and stochastic representations of the mean–eddy kinetic energy transfer term, the decorrelation timescale τ —which governs the stochastic noise and its associated covariance tensor (see Eqs.(8.21) and (8.22))—has been locally adjusted via a procedure described in Appendix 8.4. The value of τ reflects the desired degree of smoothness of the stochastic representation and is here estimated over neighborhoods of 20×20 grid points. Figure 8.4 presents the resulting map of τ .

8.3.2 Convergence of the Statistical Moments

We begin by assessing the convergence of ensemble statistics for the mean kinetic energy (K_M) and eddy kinetic energy (K_E), as illustrated in Fig. 8.5. These

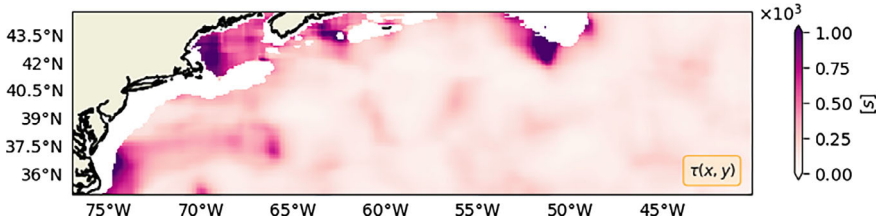


Fig. 8.4 Time scale map τ . This time scale is computed to align the amplitude of $DADV$ and MEC (8.19) (Appendix 8.4). The time scale is piecewise constant in space and has been computed over local neighborhoods of 20 by 20 grid points

diagnostics show that ensemble-averaged values of K_M and K_E stabilize after approximately 20 and 30 members, respectively. The analysis is repeated for four different dates to account for potential seasonal variability. While some minor fluctuations are observed, all curves exhibit convergence, confirming that the 48-member ensemble is sufficient for reliable estimation of K_M and K_E . However, achieving convergence for more complex diagnostics, such as the eddy–mean flow energy transfer terms, is substantially more difficult. As we will show, adopting a stochastic framework significantly improves this convergence behavior, in comparison to direct empirical statistics performed with a small ensemble.

In nonlinear systems governed by the primitive equations, the underlying attractor is typically high-dimensional and complex. Consequently, the associated statistical distributions often depart from Gaussianity. Capturing such non-Gaussian features—including skewness (asymmetry) and kurtosis (heavy tails)—requires the analysis of higher-order statistical moments. Robust estimation of these moments is thus essential for accurately characterizing variability and uncertainty.

Figure 8.6 presents a comparison of the convergence behavior of the first four statistical moments—mean, variance, skewness, and kurtosis—for both the deterministic MEC term and its stochastic counterpart, $DADV$. For MEC , statistics are computed directly from progressively larger subsets of the ensemble of 48 members. For $DADV$, convergence is evaluated by increasing the number of stochastic samples drawn from the underlying random velocity field. This field is reconstructed using the full set of empirical orthogonal functions (EOFs) obtained from the ensemble, thereby preserving all the original statistical information.

To test the robustness of the stochastic approach, four different sizes of EOF bases are considered ($N_m = 12, 24, 36$, and 48). In all cases, the statistics of $DADV$ exhibit reliable convergence, confirming the method still remains consistent for a restricted number of modes.

The results show that, even with 48 members, the MEC moments remain noisy, with convergence errors on the order of 5%. In contrast, the stochastic method achieves full convergence for the first two moments, and provides more accurate estimates for skewness and kurtosis than the deterministic approach. Beyond the mean error level, the slope of convergence—as measured by the RMSE versus the

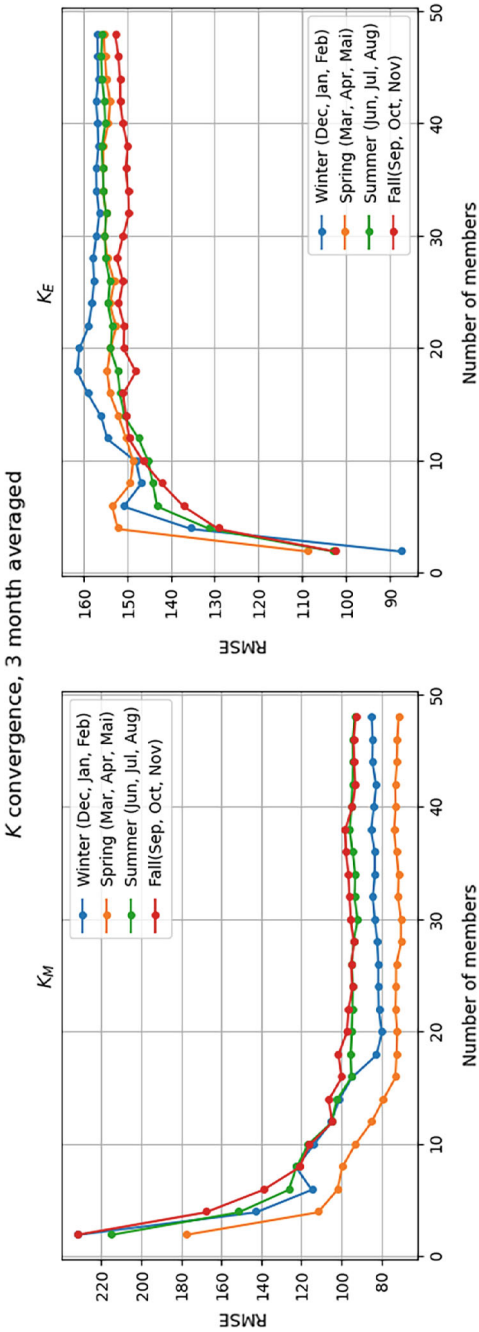


Fig. 8.5 Convergence of ensemble estimates of mean kinetic energy (K_M) and eddy kinetic energy (K_E). RMS values are computed for an increasing number of members, averaged horizontally over the domain (35°N–44°N; 75°W–40°W) at $z = 94$ m, for January 1, 2010

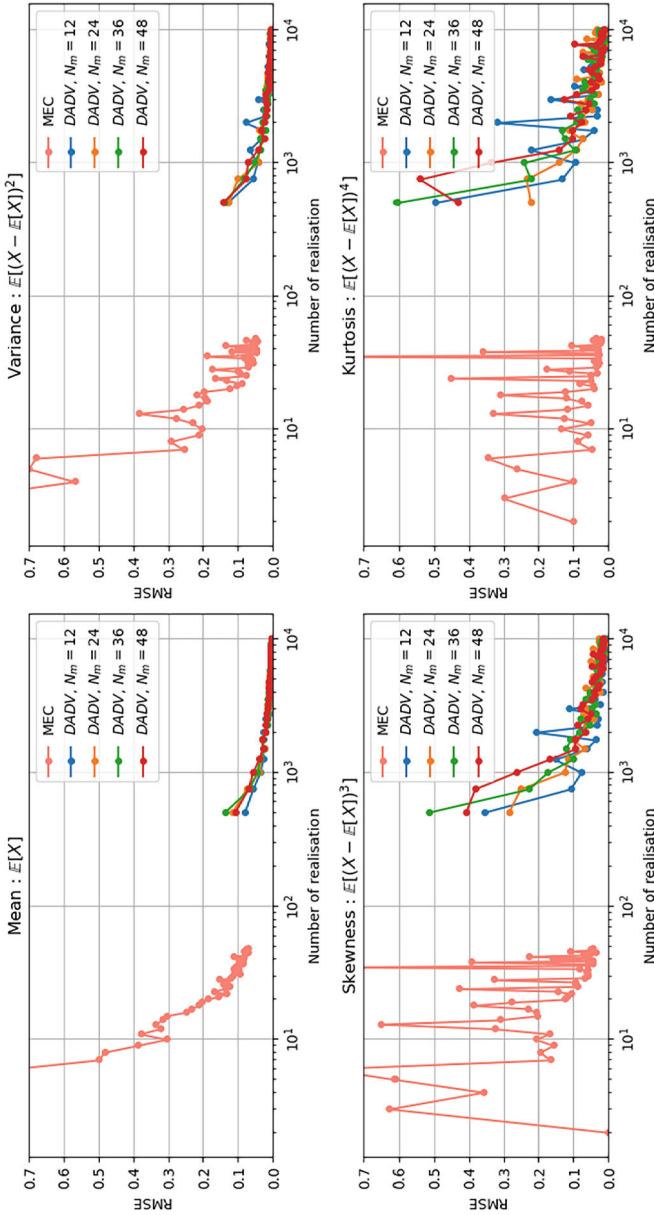


Fig. 8.6 Convergence of the first four statistical moments (mean, variance, skewness, and kurtosis) for the stochastic (*DADV*) and deterministic (*MEC*) terms. The vertical axis represents the absolute RMSE of each moment. For *MEC*, the number of ensemble members ranges from 2 to 48. For *DADV*, the number of stochastic realizations ranges from 250 to 10,000, and results are shown for four different EOF basis sizes: $N_m = 12, 24, 36$, and 48 (full set). The x-axis is in logarithmic scale. All diagnostics are computed over the domain (35°N – 44°N , 75°W – 40°W) at $z = 94$ m on January 1, 2010, with horizontal averaging

number of samples—indicates that the *MEC* statistical moments are still far from being fully converged.

From a computational standpoint, the stochastic method is highly efficient: random samples can be generated on demand, requiring storage only for the EOF basis. By comparison, improving the deterministic diagnostics would necessitate significantly larger ensembles, incurring substantial computational and storage costs.

Figure 8.7 displays the spatial structure of the first four moments of the *DADV* term. These diagnostics highlight key regions of variability and intermittency, notably near the Gulf Stream detachment and around 65°W. In these areas, pronounced skewness and kurtosis reveal significant departures from Gaussianity, indicating the presence of extreme and intermittent energy transfer events.

8.3.3 Temporal Evolution of Energy Transfer Terms

Figure 8.8 presents the temporal evolution of the deterministic *MEC* and stochastic *DADV* energy transfer terms, integrated over three spatially distinct regions along the Gulf Stream: the detachment zone (blue), a downstream segment (green), and the tail of the current (red). In the detachment region, both diagnostics exhibit a relatively steady and consistent energy loss from the mean flow, with *DADV* and *MEC* closely overlapping. This agreement suggests that the deterministic *MEC* term is well converged in this specific region, and both frameworks yield similar interpretations. Downstream (green box), the behavior of the two terms begins to diverge. The *DADV* diagnostic indicates a persistent energy gain, while the *MEC* term becomes unstable, oscillating between positive and negative values with no clear trend. In the far downstream region (red box), *DADV* again points to a consistent energy loss, whereas *MEC* remains erratic and unstable.

These oscillations in the *MEC* diagnostic seem to indicate a lack of statistical convergence, particularly in the downstream regions, suggesting that they may stem from numerical noise or sampling artifacts rather than genuine physical variability. In contrast, the *DADV* term, derived within the Location Uncertainty framework, exhibits a clearer spatial delineation between regions of energy gain and loss. Nevertheless, the time series of energy transfer across all three regions remains highly variable. Despite the improved statistical sampling enabled by the stochastic approach, the underlying dynamics of eddy–mean flow energy exchanges are still not fully understood. Further research is required to evaluate the robustness of LU-based diagnostics and to establish whether they can provide a reliable and interpretable representation of these complex interactions.

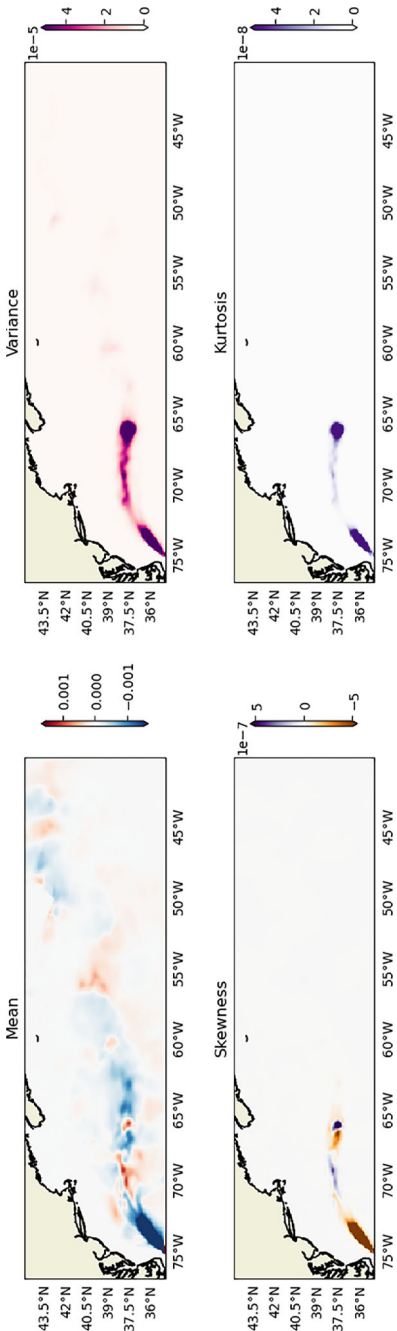


Fig. 8.7 Spatial distribution of the first four statistical moments (mean, variance, skewness, kurtosis) of the *DADV* term on January 1, 2010

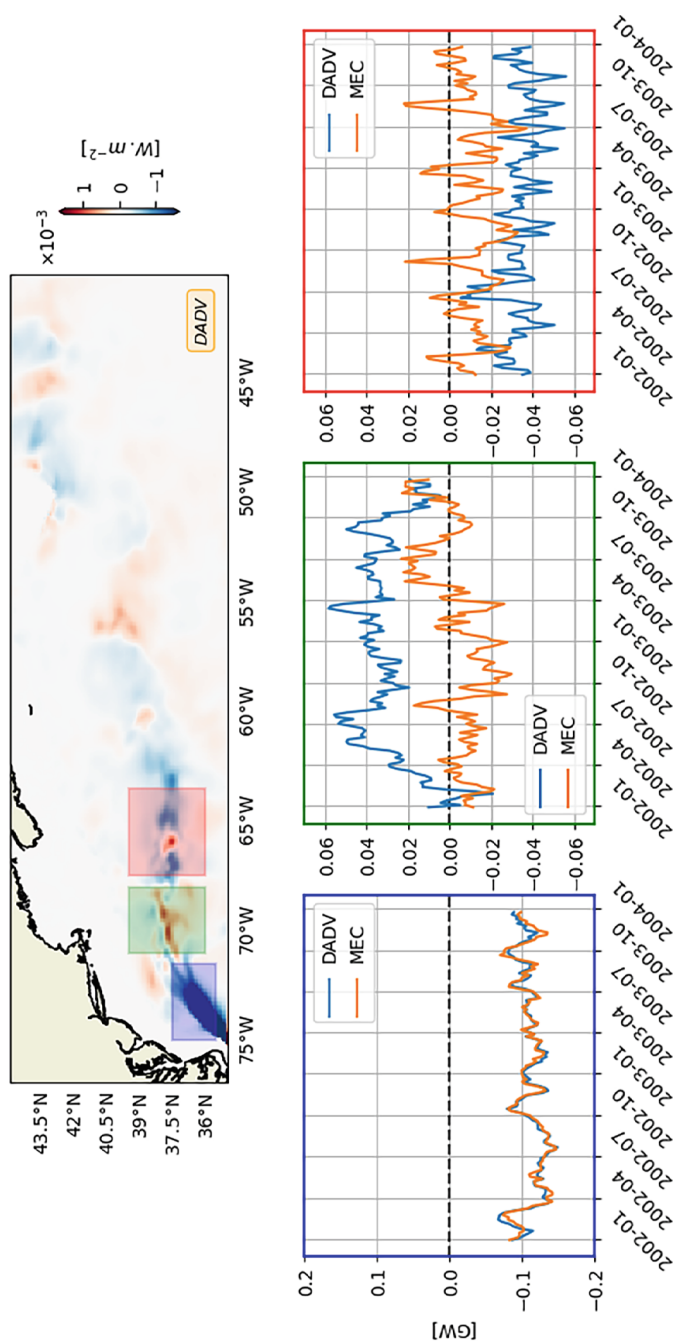


Fig. 8.8 Time series of the *DADV* (stochastic) and *MEC* (deterministic) terms, spatially integrated over three subregions; near the Gulf Stream detachment (blue), a downstream region (green), and the far tail of the current (red)

8.4 Conclusion

In this study, we investigated eddy–mean flow interactions in large-scale ocean circulation, focusing on the limitations of ensemble-based Reynolds averaging in General Circulation Models (GCMs). To address these challenges, we employed the Location Uncertainty (LU) framework, introducing a stochastic approach that enhances the statistical representation of kinetic energy exchanges between mean and eddy components.

Our results demonstrate that the stochastic formulation enables more robust and statistically convergent diagnostics of the Reynolds stress work. Specifically, the classical computation of the mean–eddy coupling term (MEC) using a 48-member ensemble suffers from significant convergence issues, limiting its reliability for energy transfer diagnostics. In contrast, the LU-based stochastic approach reconstructs eddy fluctuations through a reduced-order model—here based on Proper Orthogonal Decomposition (POD)—while preserving the full statistical information contained in the original ensemble. By incorporating stochastic diffusion and the Itô–Stokes drift, this method offers a physically consistent representation of eddy–mean flow interactions diagnostic.

The analysis of statistical moments further highlights the advantages of the LU approach, especially for higher-order metrics like skewness and kurtosis, which are crucial for capturing the non-Gaussian characteristics of energy transfer. A key strength of this framework is its ability to generate large ensembles at minimal additional computational cost, thereby improving statistical convergence and enhancing the accuracy of diagnostics. Spatial and temporal analyses reveal that energy exchanges are highly localized, exhibiting intense and coherent transfer patterns—for example, at the Gulf Stream detachment. The LU-based diagnostic, *DADV*, captures these localized phenomena more distinctly than its deterministic counterpart, despite persistent strong temporal variability. Moreover, these regions display heavy-tailed distributions, indicating the presence of rare and extreme events that deviate from classical Gaussian assumptions.

Looking ahead, future work will focus on refining the stochastic noise parameterization, investigating a wider range of decorrelation timescales, and extending the LU framework to additional energy transfer diagnostics. While this study concentrated on the mean-to-eddy energy transfer via Reynolds stress work (*MEC*), other critical processes—such as those involved in the eddy kinetic energy budget—warrant further investigation, including both local and non-local contributions (Jamet et al. 2022). A natural progression will be to develop a consistent stochastic definition of eddy kinetic energy, which will facilitate the characterization of further transfer mechanisms. By improving statistical sampling and capturing complex flow variability, the LU framework provides a promising avenue toward more accurate and physically consistent diagnostics of energy transfers in ocean circulation models.

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Appendix A: Proper Orthogonal Decomposition (POD)

Proper Orthogonal Decomposition (POD) is a widely used technique in fluid dynamics for extracting a set of orthogonal modes that optimally represent the dataset's variance (Holmes et al. 1996). As a dimensionality-reduction method, POD identifies patterns that capture the energy-dominant features of the data. It achieves this by decomposing the dataset into a set of orthogonal basis functions (modes) modulated by corresponding temporal coefficients. These modes are ranked by the amount of variance they explain in the data, enabling the construction of a reduced-order representation by retaining only the most significant modes. The decomposition works on a set of velocity fluctuation $\mathbf{u}' = (u', v')$ organized in a matrix $\mathbf{U} \in \mathbb{R}^{n \times m}$, where n is twice the number of spatial points (to include both u' and v') and m is the number of snapshots. Typically, these snapshots are part of a time series, but in this work, the decomposition is carried across the ensemble dimension. This means we work at a fixed time with $m = 48$, corresponding to the 48 ensemble members. Each column of $\mathbf{U}$ contains the velocity field $\mathbf{u}'_i \in \mathbb{R}^n$ for a specific ensemble member. Thus, the data matrix is formed as

$$\mathbf{U} = [\mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_m] \in \mathbb{R}^{n \times m}. \quad (8.25)$$

The “direct” method involves then computing the spatial covariance matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$ of the data and solving the associated eigenvalue problem:

$$\mathbf{C} = \frac{1}{m-1} \mathbf{U}^T \mathbf{U}, \quad \mathbf{C} \phi = \lambda \phi. \quad (8.26)$$

This method is efficient for velocity field of small dimension. Once the eigenvectors ϕ are computed, the original data can be projected onto the new basis to compute the POD coefficients:

$$\mathbf{A} = \mathbf{U} \phi. \quad (8.27)$$

When the number of snapshots m (or the ensemble dimension here) is much smaller than the number of spatial points n , the snapshot method (Sirovich 1987; Wang et al. 2016) is preferred. It reduces computational intensity by constructing the covariance matrix in the snapshot (ensemble) dimension:

$$\mathbf{C}_s = \frac{1}{m-1} \mathbf{U} \mathbf{U}^T. \quad (8.28)$$

The eigenvalue problem and the spatial modes are derived as

$$\mathbf{C}_s \mathbf{A}_s = \lambda \mathbf{A}_s, \quad \phi_s = \mathbf{U}^T \mathbf{A}_s. \quad (8.29)$$

Although the snapshot method is computationally faster, it introduces a normalization factor between modes and coefficients. To ensure equivalence with the direct method, normalization is applied:

$$\phi = \frac{\phi_s}{\|\phi_s\|}, \quad \mathbf{A} = \|\phi_s\| \mathbf{A}_s. \quad (8.30)$$

The system can be approximately reconstructed using a truncated sum of the most significant POD modes. The reconstructed velocity field $\tilde{\mathbf{u}}$ is then given by

$$\tilde{\mathbf{u}}(\mathbf{x}, t) \approx \sum_{k=1}^{N_{modes}} A_k(t) \phi_k(\mathbf{x}) = \sum_{k=1}^{N_{modes}} A_{s,k}(t) \phi_{s,k}(\mathbf{x}). \quad (8.31)$$

Degree of approximation of the flow can be estimated by comparing the eddy kinetic energy over the domain as $EKE = \int \mathbf{u}' \cdot \mathbf{u}' d\mathbf{x}$ and compare it to the sum of eigenvalues $\sum \lambda_k$, which represents the energy of the approximation.

Appendix B: Stochastic Evolution Equation for K_M

$$\begin{aligned} \rho_0 \overline{\mathbf{u}_h} \cdot \overline{d\mathbf{u}_h} &= \rho_0 \overline{\mathbf{u}_h} \cdot \overline{\left(-\nabla \cdot (\mathbf{u}_h \otimes (\mathbf{u}^* dt + \sigma d\mathbf{B}_t)) + \frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \mathbf{u}_h) dt - f \mathbf{k} \times \mathbf{u}_h dt - \frac{1}{\rho_0} \nabla dP \right)} \\ d_t K_M &= -\rho_0 \overline{\mathbf{u}_h} \cdot \overline{\left(\nabla \cdot (\mathbf{u}_h \otimes (\mathbf{u}^* dt + \sigma d\mathbf{B}_t)) \right)} + \rho_0 \overline{\mathbf{u}_h} \cdot \frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \overline{\mathbf{u}_h}) dt \\ &\quad - \underbrace{\overline{\rho_0 \mathbf{u}_h \cdot f \mathbf{k} \times \mathbf{u}_h}}_{=0} dt - \overline{\mathbf{u}_h} \cdot \nabla d\overline{P} \\ d_t K_M &= -\rho_0 \overline{\mathbf{u}_h} \cdot \overline{\left(\nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}^*) dt + \nabla \cdot (\underbrace{\mathbf{u}_h \otimes \sigma d\mathbf{B}_t}_{=0}) \right)} + \rho_0 \overline{\mathbf{u}_h} \cdot \left(\frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \overline{\mathbf{u}_h}) dt \right) \\ &\quad - \nabla \overline{\mathbf{u}} \overline{P} dt - \overline{w} \overline{b} dt \end{aligned} \quad (8.32)$$

With all the terms in $d\mathbf{B}_t$ cancelling in mean, we obtain the time derivative $\partial_t K_M$:

$$\partial_t K_M = -\rho_0 \overline{\mathbf{u}_h} \cdot (\nabla \cdot (\overline{\mathbf{u}_h} \otimes \overline{\mathbf{u}})) + \rho_0 \overline{\mathbf{u}_h} \cdot (\nabla \cdot (\overline{\mathbf{u}_h} \otimes \mathbf{u}_s)) + \rho_0 \overline{\mathbf{u}_h} \cdot \left(\frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \overline{\mathbf{u}_h}) \right) - \nabla \overline{\mathbf{u}} \overline{P} - \overline{w} \overline{b}, \quad (8.33)$$

which simplifies as

$$\partial_t K_M = -\nabla \cdot (\overline{\mathbf{u}} K_M) + \nabla \cdot (\mathbf{u}_s K_M) + \rho_0 \overline{\mathbf{u}_h} \cdot \left(\frac{1}{2} \nabla \cdot (\mathbf{a} \nabla \overline{\mathbf{u}_h}) \right) - \nabla \overline{\mathbf{u}} \overline{P} - \overline{w} \overline{b}. \quad (8.34)$$

Appendix C: Milstein-Type Scheme for Transport Noise

In this section, we describe the time-stepping scheme in the stochastic case, focusing on the noise advection and diffusion terms in the stochastic transport equation for a scalar quantity q (i.e., the resolved advection velocity, $\mathbf{u}^*$, is here assumed to be null without loss of generalization). In compact form, the equation is expressed as follows:

$$d_t q = f(q, t) dt + \sum_{i=1}^m g^i(q, t) d\beta_t^i, \quad (8.35)$$

$$g^i(q, t) = -\varphi_t^i \cdot \nabla q, \quad (8.36)$$

$$f(q, t) = \frac{1}{2} \nabla \cdot (\mathbf{a}_t \nabla q) = \frac{1}{2} \sum_{i=1}^m g^i \circ g^i(q, t), \quad (8.37)$$

where $(\varphi_t^i)_{i=1,\dots,m}$ represents a set of non-divergent ($\nabla \cdot \varphi_t^i = 0$) basis functions at time t , and $(\beta^i)_{i=1,\dots,m}$ is a sequence of independent standard Brownian motions. The last equality in (8.37) derives from the definition of the diffusion tensor, $\mathbf{a}_t = \sum_{i=1}^m \varphi_t^i (\varphi_t^i)^\top$, along with the divergence-free property of φ_t^i . We now consider the semi-discrete form of this equation:

$$dq_h(t) = f_h(q_h(t), t) dt + \sum_{i=1}^m g_h^i(q_h(t), t) d\beta_t^i. \quad (8.38)$$

Here, the subscript h denotes quantities or operators that have been discretized in space. Applying the functional extension of the Itô formula (Cont and Fournie 2010) to both f_h and g_h^i , we obtain

$$\begin{aligned} q_h(t) = q_h(s) &+ \int_s^t f_h(q_h(r), r) dr + \int_s^t \sum_{i=1}^m g_h^i(q_h(r), r) d\beta_r^i \\ &+ \int_s^t \sum_{j=1}^m \dot{g}_h^j \left(\int_s^r \sum_{i=1}^m g_h^i(q_h(p), p) d\beta_p^i, r \right) d\beta_r^j + R, \end{aligned} \quad (8.39)$$

where R denotes higher-order terms in the Milstein scheme, and $\dot{g}_h^i$ denotes the derivative of g_h^i with respect to function q_h . From (8.36), we find that $\dot{g}_h^i = -\varphi^i \cdot \nabla = g_h^i$. Consequently, we obtain the following first-order approximation of the above integral problem:

$$\begin{aligned}
q_h^{n+1} = q_h^n &+ f_h(q_h^n, t_n) \Delta t + \sum_{i=1}^m g_h^i(q_h^n, t_n) \Delta \beta^i \\
&+ \sum_{i,j=1}^m g_h^j \circ g_h^i(q_h^n, t_n) \underbrace{\int_{t_n}^{t_{n+1}} \int_{t_n}^r d\beta_p^i d\beta_r^j}_{:=I_{ij}}, \tag{8.40}
\end{aligned}$$

where q^n is the value of q at time t_n , $\Delta t = t_{n+1} - t_n$ is the time step, and $\Delta \beta \sim \sqrt{\Delta t} \mathcal{N}(0, 1)$ represents the Brownian motion increment. The double integral I_{ij} can be decomposed into a symmetric and an anti-symmetric part:

$$I_{ij} = \frac{1}{2} \underbrace{(I_{ij} + I_{ji})}_{:=S_{ij}} + \frac{1}{2} \underbrace{(I_{ij} - I_{ji})}_{:=A_{ij}}, \tag{8.41}$$

where the symmetric component expands as follows:

$$\begin{aligned}
S_{ij} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^s d\beta_r^i d\beta_s^j + \int_{t_n}^{t_{n+1}} \int_{t_n}^s d\beta_r^j d\beta_s^i \\
&= \int_{t_n}^{t_{n+1}} \beta_s^i d\beta_s^j + \int_{t_n}^{t_{n+1}} \beta_s^j d\beta_s^i - \int_{t_n}^{t_{n+1}} \beta_{t_n}^i d\beta_s^j - \int_{t_n}^{t_{n+1}} \beta_{t_n}^j d\beta_s^i \\
&= \beta_{t_{n+1}}^i \beta_{t_{n+1}}^j - \beta_{t_n}^i \beta_{t_n}^j - \delta_{ij} \Delta t - \beta_{t_n}^i \beta_{t_{n+1}}^j + \beta_{t_n}^j \beta_{t_n}^i - \beta_{t_{n+1}}^i \beta_{t_n}^j + \beta_{t_n}^i \beta_{t_n}^j \\
&= \underbrace{(\beta_{t_{n+1}}^j - \beta_{t_n}^j)}_{\Delta \beta^j} \underbrace{(\beta_{t_{n+1}}^i - \beta_{t_n}^i)}_{\Delta \beta^i} - \delta_{ij} \Delta t \tag{8.42}
\end{aligned}$$

using the Itô's integration-by-part formula (Kunita 1990) for two independent Brownian motions. Therefore, I_{ij} can be rewritten as

$$I_{ij} = \frac{1}{2} (\Delta \beta^i \Delta \beta^j - \delta_{ij} \Delta t + A_{ij}). \tag{8.43}$$

Substituting this into (8.40) and recalling (8.37), we derive the Milstein scheme for the semi-discrete problem (8.38):

$$q_h^{n+1} = q_h^n + \sum_{i=1}^m g_h^i(q_h^n, t_n) \Delta \beta^i + \frac{1}{2} \sum_{i,j=1}^m g_h^j \circ g_h^i(q_h^n, t_n) (\Delta \beta^j \Delta \beta^i + A_{ij}). \tag{8.44}$$

Accordingly, the diffusion term in the continuous formulation is precisely balanced by the diagonal component of the higher-order double integral. The anti-symmetric component, A_{ij} , known as the Lévy area process, is notably challenging to simulate. However, as numerically supported in recent studies (Boulevard and Mémin 2023; Fiorini et al. 2023), for LU types dynamics, the Milstein scheme, without Lévy

area, significantly reduces approximation error relative to the Euler scheme, though the strong convergence order remains the same. Consequently, we choose to disregard A_{ij} in practical applications. Additionally, as illustrated in Mémin et al. (2024), for linear waves, the proposed scheme enables to implement effectively the energy balance between the noise and the diffusion through an implicit discretization of the anisotropic diffusion operator involved (via the iterative advection term).

With the discrete noise term given by $\sigma^n \Delta \mathbf{B} = \sum_{i=1}^m \varphi_{t_n}^i \Delta \beta^i$, generated at each time step, the previous scheme can be expressed in a more compact form as follows:

$$q_h^{n+1} = q_h^n - (\sigma^n \Delta \mathbf{B} \cdot \nabla q^n)_h + \frac{1}{2} (\sigma^n \Delta \mathbf{B} \cdot \nabla (\sigma^n \Delta \mathbf{B} \cdot \nabla q^n))_h. \quad (8.45)$$

Incorporating transport noise into the Leapfrog scheme used in the current version of NEMO, we propose the following multi-scale approach:

$$\begin{aligned} \tilde{q}_h &= q_h^{n-1} + F_h(q_h^n, t_n) \Delta t \\ q_h^{n+1} &= \tilde{q}_h + F_h(q_h^n, t_n) \Delta t - (\sigma^n \Delta \mathbf{B} \cdot \nabla q^n)_h + \frac{1}{2} (\sigma^n \Delta \mathbf{B} \cdot \nabla (\sigma^n \Delta \mathbf{B} \cdot \nabla q^n))_h. \end{aligned} \quad (8.46)$$

In this scheme, F gathers the classical resolved advection together with the advection of q by the Itô–Stokes drift $\mathbf{v}_s = \frac{1}{2} \nabla \cdot \mathbf{a}$ in the stochastic formulation. This approach is also adaptable to other schemes, such as Adams–Bashforth or Runge–Kutta, widely used in OGCMs, where transport noise would be included in the final correction.

Appendix D: Decorrelation Timescale τ

A decorrelation timescale τ is required for an accurate representation of the energy transfer. To that end, the stochastic term should be representative of the *MEC* as written in (8.19):

$$- \rho_0 \overline{\mathbf{u}_h} \cdot \nabla \cdot (\overline{\mathbf{u}_h' \otimes \mathbf{u}_h'}) = \nabla \cdot (\mathbf{u}_s K_M) + \rho_0 \overline{\mathbf{u}_h} \cdot \mathbb{E}((\sigma d\mathbf{B}_t \cdot \nabla)(\sigma d\mathbf{B}_t \cdot \nabla) \overline{\mathbf{u}_h}),$$

which reads integrating in space:

$$\begin{aligned} \int_{\Omega} -\rho_0 \overline{\mathbf{u}_h} \cdot \nabla \cdot (\overline{\mathbf{u}_h' \otimes \mathbf{u}_h'}) dx &= \int_{\Omega} \nabla \cdot ((\frac{1}{2} \nabla \cdot \mathbf{a}) K_M) \\ &\quad + \rho_0 \overline{\mathbf{u}_h} \cdot \mathbb{E}((\sigma d\mathbf{B}_t \cdot \nabla)(\sigma d\mathbf{B}_t \cdot \nabla) \overline{\mathbf{u}_h}) dx, \end{aligned}$$

and expressed in spectral form, where the decorrelation time is explicitly involved:

$$\begin{aligned}
\int_{\Omega} -\rho_0 \overline{u_h} \cdot \nabla \cdot (\overline{u'_h \otimes u'_h}) dx &= \int_{\Omega} \nabla \cdot ((\frac{1}{2} \nabla \cdot (\sum_k \tau \phi_k \phi_k^T)) K_M) \\
&\quad + \rho_0 \overline{u_h} \cdot \mathbb{E}((\sum_k \sqrt{\tau} \phi_k dB_k \cdot \nabla)(\sum_k \sqrt{\tau} \phi_k dB_k \cdot \nabla) \overline{u_h}) dx.
\end{aligned} \tag{8.47}$$

To infer a local expression of the time decorrelation, we consider any neighborhood $\nu(\mathbf{x})$ centered at (x, y) , within which τ is assumed constant. This allows us to compute a piecewise approximation of $\tau(x, y)$, yielding the following expression for the timescale:

$$\tau(\mathbf{x}) = A^{-1} \int_{\nu(\mathbf{x})} -\rho_0 \overline{u_h} \cdot \nabla \cdot (\overline{u'_h \otimes u'_h}) d\mathbf{x}, \tag{8.48}$$

where the factor reads

$$A = \int_{\nu(\mathbf{x})} \nabla \cdot ((\frac{1}{2} \nabla \cdot (\sum_k \phi_k \phi_k^T)) K_M) + \rho_0 \overline{u_h} \cdot \mathbb{E}((\sum_k \phi_k dB_k \cdot \nabla)(\sum_k \phi_k dB_k \cdot \nabla) \overline{u_h}) d\mathbf{x}. \tag{8.49}$$

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