# Optimal surface salinity perturbations of the Meridional Overturning Circulation

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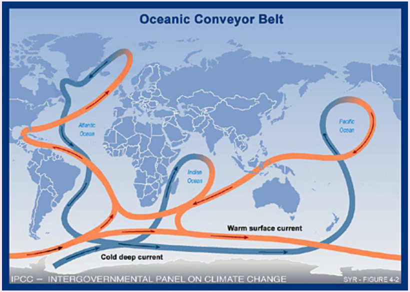
EGU
April 2009

# Climate context (1) : The MOC

Slow dynamics of the ocean :

Meridional Overturning Circulation (MOC)

- Intensity of $\sim$18 Sv
- Time scale of $\sim$500 ans




Northward transport of heat influencing the European climate

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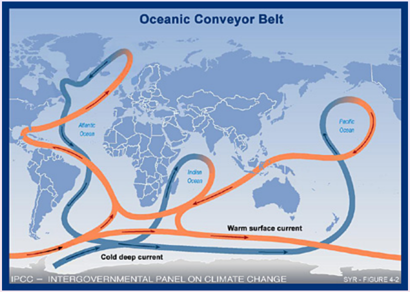


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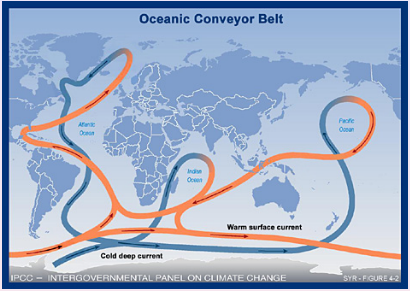


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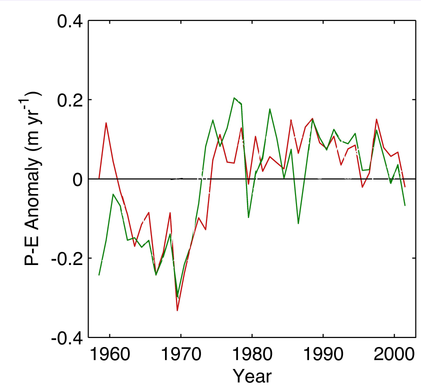
# Climate context (2) : North Atl. P-E

- Increase of precipitation in the north Atlantic

  → Josey and Marsh (2005)

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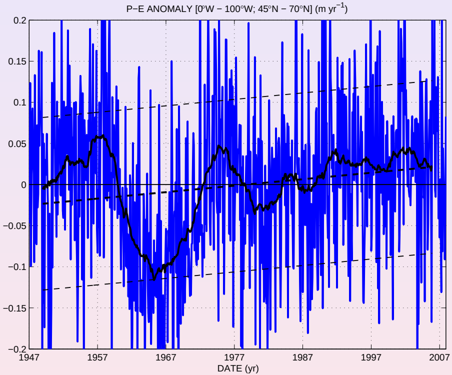


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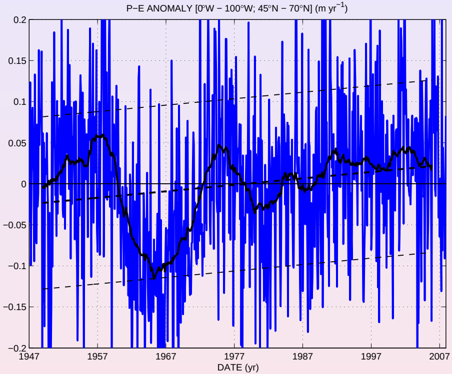


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# Approach

⇒Forced variability of the ocean circulation

- Linear approach :
  weak variations (perturbations) of the ocean circulation
- Generalized stability analysis :
  - Atmosphere ⇒ optimal initial and stochastic perturbation (Farrell and Ioannou, 1996)
  - Ocean ⇒ optimal initial perturbation (Moore and Farrell, 1993)
  - 3 box THC ⇒ optimal initial and stochastic perturbation (Tziperman and Ioannou, 2002)

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# Maximization method : Oceanic circulation application

Goal :

- Optimal impact of the SSS on the circulation

Lagrange parameters method

- Functions to maximize $\langle F|u(t)\rangle$ (or $\langle F|u(t)\rangle^2$) :
  - Meridional Overturning Circulation (MOC) at the latitude and depth of its steady state maximum (or its variance)
- Constraints
  1. Normalisation : $\langle u(0)|\mathbf{S}|u(0)\rangle = 1$
  2. Salt conservation : $\langle C|u(0)\rangle = 0$
  3. Only surface salinity perturbation : $|u(0)\rangle = \mathbf{P}\,|u'\rangle$

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# Optimal perturbation experiments

| | 2D | PG | OPA - ORCA2 |
|---|---|---|---|
| Initial perturbation | X | X | X |
| Constant perturbation | X | | |
| Stochastic perturbation | X | X | |

- Latitude-depth model
  ⇒ Methodological study
  Sévellec et al. (*J. Phys. Oceanogr.*, 2007)
- Planetary geostrophic model :
  ⇒ Influence of the surface boundary condition (flux *vs* mixed)
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# Upper bound in the 2D model

- Initial SSS perturbation :
  Great Salinity Anomalies (GSA, Belkin et al., 1998)
  0.5 psu on 250 m $\Rightarrow$ 2 Sv
- Constant FW perturbation :
  hydrological cycle modification in the global warming scenario (Held and Soden, 2006)
  4% (3 cm yr$^{-1}$) $\Rightarrow$ 0.14 Sv
- Stochastic FW perturbation :
  Using 2 different reanalysis datasets
  5 cm yr$^{-1}$ $\Rightarrow$ 4.6 Sv

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# Results in the PG model

- Variability :
  - Large scale gradient SSS efficiently stimulates a North Atl. multidecadal oscillation.
- Surface boundary condition :
  - The sensitivity pattern weakly depends of the surface boundary condition.
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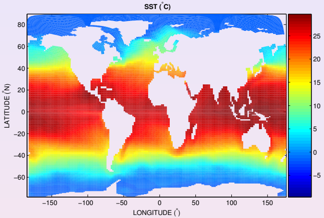
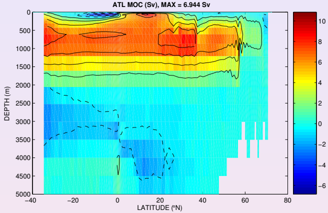
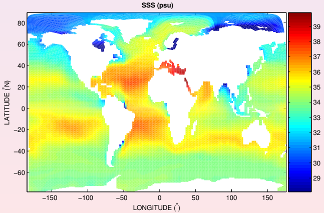
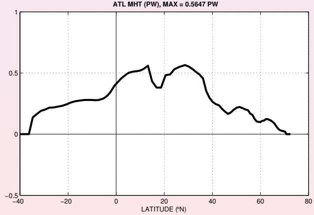
# Application in an Ocean General Circulation Model

Model : OPA 8.2, ORCA2, OPATAM



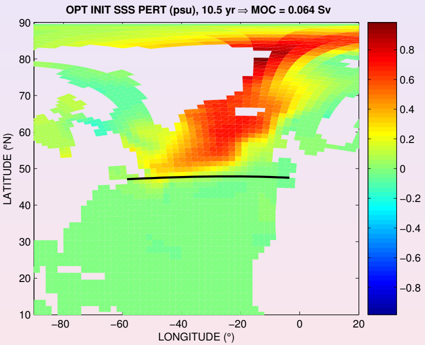



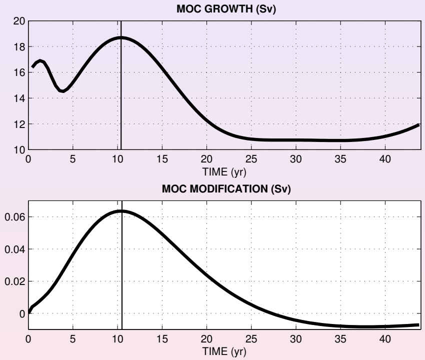



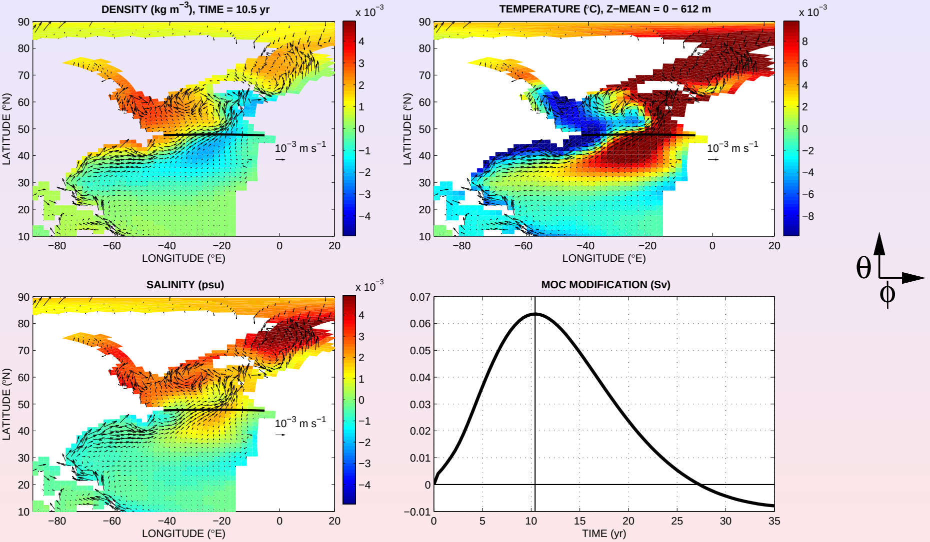

- MAX(MOC)=7 Sv (48°N)
- MAX(MHT)=0.6 PW (27°N)

# Optimal initial SSS perturbation for the MOC



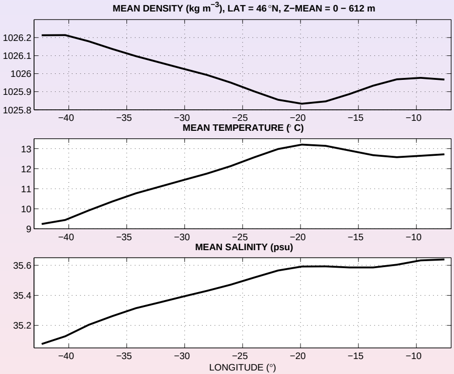




- Maximum growth after 10.5 yr




- $v'_{\mathrm{surf}} > 0$
- $|\beta \partial_\theta S'| \gg |\alpha \partial_\theta T'|$
- $|\alpha \partial_\phi T'| \gg |\beta \partial_\phi S'|$

# Finite time growth mechanism




$$\alpha\partial_\phi \bar{T} \gg \beta\partial_\phi \bar{S}$$

$$\mathrm{SSS}'_{\mathrm{north}} > 0 \Rightarrow v'_{\mathrm{surf}} > 0$$

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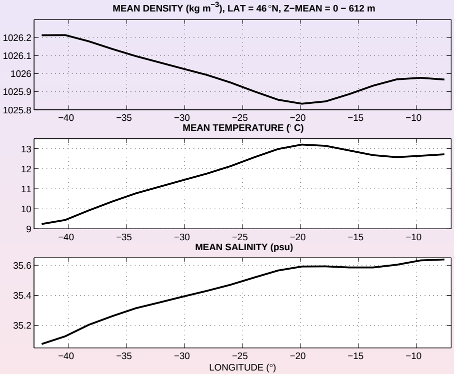


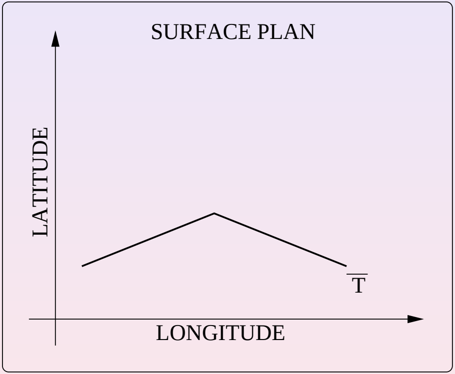


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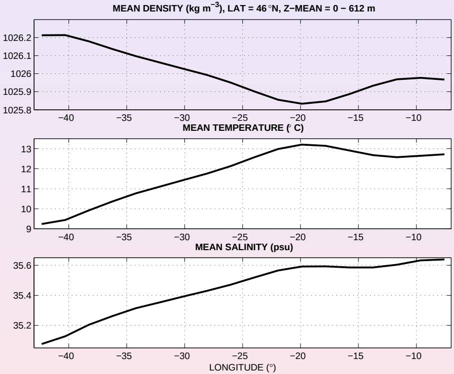


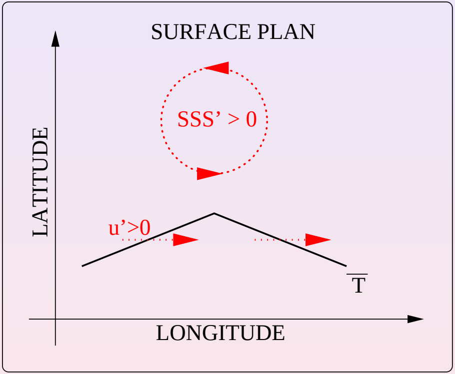


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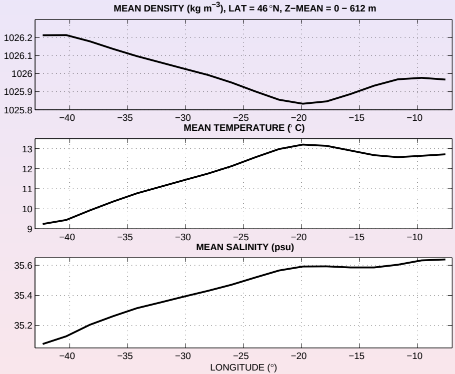


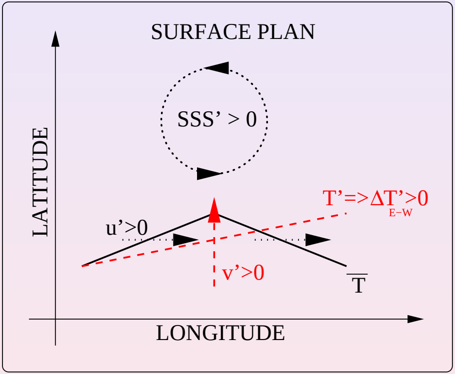


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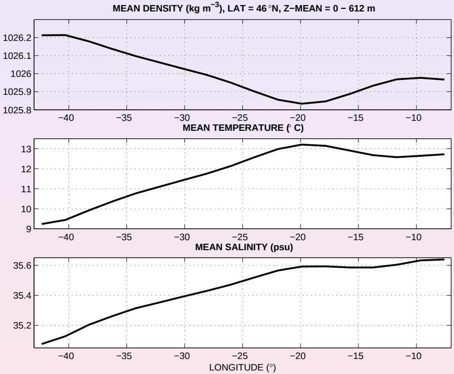


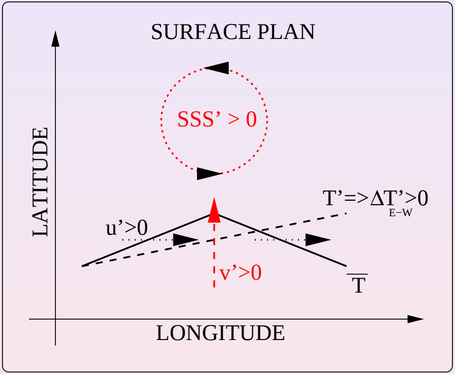


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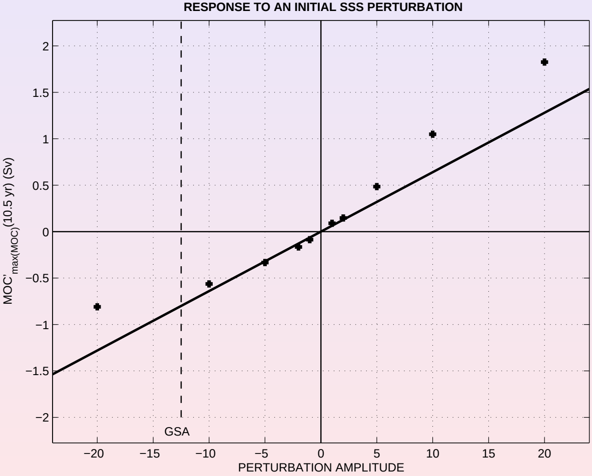


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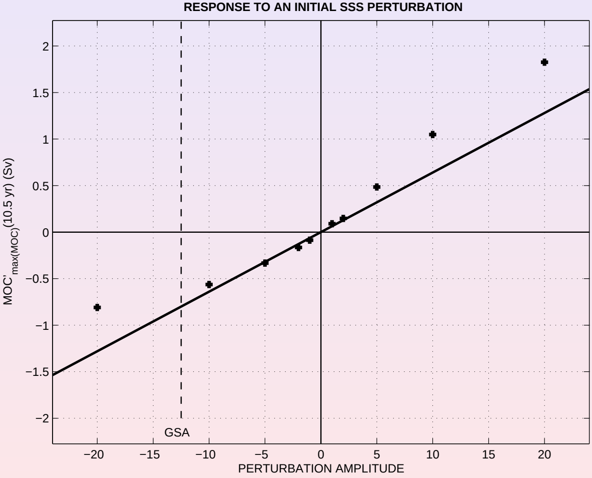


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# Results

- Efficient method to obtain the optimal initial perturbation :
  $\Rightarrow$ Explicit solution (adj. model)
- Results of the 2D, PG and OGCM models
  $\Rightarrow$ Similarity :
  - Optimal pert. are large scale North-South gradient
  - In 3D, the sensitivity is dominated by the Salinity
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  $\Rightarrow$ Difference : Transient growth mechanism
- Optimal SSS perturbation of the MOC in an OGCM
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# Future work

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  - Impact of the Southern Ocean
  - Mechanism of the finite time growth
- Seasonal cycle (non-autonomous operator)
  - Sensitivity to the season
- Tropical study :
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Thank you for your attention

# Optimal initial SSS perturbation

Perturbation evolution (autonomous problem) :

$$\partial_t \left| u \right\rangle = \mathbf{A} \left| u \right\rangle ,$$

$$\Rightarrow \left| u(\tau) \right\rangle = \mathbf{M}(\tau) \left| u(0) \right\rangle = e^{\mathbf{A}\tau} \left| u(0) \right\rangle .$$

Explicit solution (using the adjoint model) of the optimal initial perturbation :

$$\Rightarrow \left| u(0) \right\rangle = \mathbf{P} \left| u' \right\rangle$$

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$\Rightarrow$ Solution depends on the maximization delay $\tau$

# Efficient method :

Maximization under constraints : $dG(\gamma, |u_0\rangle) = 0$

- Measure : Linear function

$$G(\gamma, |u_0\rangle) = \langle F|\mathbf{M}(\tau)|u_0\rangle - \gamma(\langle u_0|\mathbf{S}|u_0\rangle - 1)$$

Explicit soluton :

$$|u_0\rangle = \pm \frac{\mathbf{S}^{-1}\mathbf{M}^{\dagger}(\tau)\,|F\rangle}{\sqrt{\langle F|\mathbf{M}(\tau)\mathbf{S}^{-1}\mathbf{M}^{\dagger}(\tau)|F\rangle}}$$

- Measure : quadratic norm

$$G(\gamma, |u_0\rangle) = \langle u_0|\mathbf{M}^{\dagger}(\tau)|\mathbf{S}_2|\mathbf{M}(\tau)|u_0\rangle - \gamma(\langle u_0|\mathbf{S}_1|u_0\rangle - 1)$$

Eigenvalue solution :

$$\gamma\mathbf{S}_1\,|u_0\rangle = \mathbf{M}^{\dagger}(\tau)\mathbf{S}_2\mathbf{M}(\tau)\,|u_0\rangle,\ \langle u_0|\mathbf{S}_1|u_0\rangle = 1$$

# Efficient method :

Maximization under constraints : $dG(\gamma, |u_0\rangle) = 0$

- Measure : Linear function

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Explicit soluton :

$$|u_0\rangle = \pm \frac{\mathbf{S}^{-1}\mathbf{M}^{\dagger}(\tau)\,|F\rangle}{\sqrt{\langle F|\mathbf{M}(\tau)\mathbf{S}^{-1}\mathbf{M}^{\dagger}(\tau)|F\rangle}}$$

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